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THE EQUATIONS UNVEILED: SRAFFA'S PRICE EQUATIONS IN THE MAKING

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In the first part of *Production of Commodities by Means of Commodities*, four equation systems are introduced: three are drawn up in order to solve the problem of relative prices; the last one is devised to define a suitable standard of prices. The book was published in 1960, but—as we are told in the preface—its 'central propositions' and a first version of the price equations had been originally conceived and written more than 30 years before, when the author was still in his twenties. Having access now to the Sraffa Papers, preserved in the Wren Library, we can ascertain the intellectual origin of the equations. In this paper the analytical path that led to the final draft of the price equation is followed, step by step, and the link between these equations and Sraffa's quest for an 'invariable standard of value' is clarified.

> The significance of the equations is simply this: that if a man fell from the moon on the earth, and noted the amount of things consumed in each factory and the amount produced by each factory during a year, he would deduce at which values the commodities must be sold, if the rate of interest must be uniform and the process of production repeated. In short, the equations show that the conditions of exchange are entirely determined by the conditions of production. (P. Sraffa, 'Man from the Moon', D3/12/7) [1]

In the first chapters of *Production of Commodities by Means of Commodities* Sraffa introduces four equation systems: three are drawn up, in quick succession, to solve the problem of relative prices; the last one is devised to define a suitable 'standard of prices'. The book was published in 1960, but—as we are informed in the Preface—its 'central propositions', and a first version of the price equations, were originally conceived and written more than 30 years before, when the author was still in his twenties.

* The author is part of a team with the task of editing the Sraffa papers. However, the present paper was written before the team was constituted. Therefore, the opinions expressed are the author's alone and they do not necessarily represent the opinions of the scholars who are now working on the documents to which the paper makes reference. I wish here to thank Pierangelo Garegnani, Sraffa's literary executor, who allowed me to examine and quote Sraffa's papers. I also wish to thank Giancarlo de Vivo, who discussed with me at length the arguments of this essay. An Italian version was presented in Rome at Accademia dei Lincei in 2003, where I have benefited from helpful comments.

[1] References in parenthesis are to the Catalogue of *Piero Sraffa Papers,* edited by Jonathan Smith, Wren Library, Trinity College, Cambridge.

Where do these equations come from? In other words, what is their original source of inspiration? From this point of view, the book and its Appendix D (References to the literature) are not particularly helpful. Having access now to the Sraffa Papers, conserved in the Wren Library, we can suggest some reasonable answers. But new interpretative problems arise. The original draft of the 'first equations', those equations which are somewhat abruptly introduced in the very first lines of the book, appears cryptic and obscure, so obscure as to suggest an inconsistent notation. Moreover, when he eventually reached a more readable draft (with a clear specification of the unknowns) Sraffa adopted what he called his 'hypothesis'. He was so deeply attached to this hypothesis that, when he was forced to abandon it, he spoke of a 'disastro del modello'.[1] And the hypothesis amounts, apparently, to the assumption of physical homogeneity between input and net output for the whole economy. This is a most surprising and un-Sraffian assumption indeed, especially if we take into account the critiques Sraffa made in those same years to the marginalist theory of capital.

In this note we would like to: (i) clarify the origin of the 'first equations' and therefore; (ii) show that their first draft was perfectly comprehensible (if unsatisfactory, and consequently rapidly abandoned); (iii) follow, step by step, the analytical path that led to their final draft; and (iv) elucidate the link between Sraffa's 'hypothesis' and his quest for an 'invariable standard of value'.

Let us begin by clearly stating our opinion that Sraffa's source of inspiration, as far as the equations are concerned, should not be sought in Marshallian or in Ricardian theory (as is commonly maintained), but in that of Marx. And, more precisely, not in Marx's value-theory (*Capital,* Book I) nor in his price-of-production theory (*Capital,* Book III), but in the reproduction schemes of *Capital,* Book II.[2]

We intend to prove later that the reproduction schemes are the obvious starting point for the analytical path followed by Sraffa.[3] Let us now only hint at three elements of documental evidence that point in this direction. First, Sraffa consistently labels his price equations without surplus as '1st equations'. The same expression ('1st equations') can be found written in the margin of his copy of *Capital* to designate the scheme of simple reproduction. Second, in a document (D3/12/16) dated 7 August 1942, he writes: '1st equations (simple reproduction)'. Finally, a few days before (30 July 1942) Sraffa, commenting on Rosa Luxemburg's opinion, according to which the reproduction schemes can be considered as an up-to-date version of the *Tableau Économique,* writes: 'Equations = Tableau Economique'.

As is well known, and as has just been recalled, the reproduction schemes are an offshoot of the Marxian 'discovery' of the *Tableau* in 1862. This whole matter is largely independent of the rest of the analysis in *Capital.* That is to say that the use of the

[1] 'Disaster of the model'.

[2] Book II of *Capital* was at the time very little known, in England as well as in Italy (the first Italian translation is dated 1946). Sraffa consistently used the French translation published in 1900. Presumably, he bought this translation, together with that of the *Theories of Surplus Value (Histoire des doctrines)* in 1927, in Paris, during a stop in his journey to England. On the reasons which can have attracted Sraffa's attention to the reproduction schemes, see Gilibert (2001).

[3] A first, if cursory, suggestion in this direction can be found in Eatwell, Panico (1987). The suggestion was taken up, without going any deeper, by Samuelson (1990).

reproduction schemes as a starting point for Sraffa implies the adoption of a 'classical' analytical approach (the system of production viewed as a circular process) but not necessarily the adoption of a strictly Marxian approach.

In our exploration, two fundamental documents will be used as guides. The first is a working research programme, a sort of strategic plan for the future book. The second is an expositional programme, conceived in order to communicate already achieved results to the outside world: a sort of tactical plan, therefore.

The research programme can be found in a notebook dated July 1928; the expositional programme bears the date of 21 August 1942. The two documents therefore correspond to the two periods in which Sraffa focused his attention on the price equations: the period 1927–1931 and the period 1940–1943. In the remaining years up to 1955, Sraffa was entirely absorbed by the monumental edition of Ricardo's works for the Royal Economic Society.

Let us quote the two documents. The first (D 3/12/9) is reproduced here in full:

Marx, Cap. vol. 2°, cap. I-III della Pte 1ª.
Considera sempre, prima riproduz. semplice, dove il capitalista consuma tutto il plusvalore; poi riproduz. con accumulaz. di tutto il plusvalore. Io dovrò considerare:

1) riproduz. semplice senza plusvalore
2) riproduz. semplice con plusvalore tutto consumato.
   2b) uguale a 2 in forma di 1 (senza r)
3) riproduz. con accumulaz. totale (necessità di interesse su cap. fisso: se no, non è possibile accumulaz. proporzionale in tutte le industrie. Chi presterebbe a un'industria che non rende abbastanza per riprodursi? Ma le macchine usate valgon meno delle nuove: però se l'accumulaz. è avvenuta sempre nel passato, la media delle macchine è più nuova del normale e quindi riceve più ammortamento di quel che spenda: è ciò esattamente uguale al richiesto?)
4) riproduz. con accumulaz. accelerata (la percentuale accumulata aumenta ogni anno a causa di invenzioni).[1]

In fact, as we shall see, Sraffa, having come to point 3, abandoned the envisaged path.

The following is the text of the relevant parts of the second document (D3/12/16), entitled 'Crosscap':[2]

Questa manovra è il centro dell'operazione, e tutto dipende dal suo successo. Va condotta come segue.

[1] 'Marx, *Capital,* vol. 2, ch. I-III of Part I.
Marx always considers simple reproduction first, where capitalists consume their whole surplus value; then he considers reproduction with accumulation of the whole surplus value. I shall consider:

1) simple reproduction without surplus value
2) simple reproduction with surplus value entirely consumed
   2b) equal to 2 in form of 1 (without r)
3) reproduction with total accumulation (necessity of interest on fixed capital. Otherwise, proportional accumulation in every industry would be impossible. Who would lend capital to an industry which does not yield enough for its own reproduction? But used machines have lesser value than new ones. However, if there has been regular accumulation in the past, the average machine is newer than normal and therefore receives a higher amortization than it spends: is it exactly what is required?)
4) Reproduction with accelerated accumulation (the accumulated percentage increases each year because of inventions)'.

[2] The *Crosscap* is a figure of topology, obtained by attaching a disk to the edge of a Möbius strip. I owe this information to Antonella Palumbo. The omitted passages of the document regard principally the insertion of fixed capital and the necessity of avoiding 'vulgar' terminology.

Prima sviluppare le 1^e equazioni, poi le seconde (con *r*), poi introdurre in queste *w* come variabile. Qui è il punto delicato: dire il più possibile, senza dar via il segreto del rapporto costante fra C e V+S: se possibile, dire che la composizione organica (usar termini volgari) dei due gruppi è identica: e forse esaminare in dettaglio gli effetti dei cambiamenti di *r* e *w* sui prezzi di singole merci; ma in ogni caso, riservare il clou per più tardi. [...]
Qui viene il Crosscap. Esaminando il Toy III si nota un rapporto fisso (fra capitale, o parte del capitale) e prodotto che è indipendente da *r*. Evitando accuratamente di mettere in luce le altre conseguenze, accorgersi che ciò fornisce un metodo (trick) per risolvere tutte (? o almeno I-III) le equazioni precedenti. Concentrare l'attenzione su questo metodo di soluzione: possiamo dare a *r* un valore arbitrario (p. es. 10% o 5% o 0%), rendendo così le equazioni lineari, risolverle, e ottenere il "rapporto fisso", dal quale poi si deriva subito il vero *r*, e finalmente si possono risolvere le equazioni reali. Con questo metodo passare di nuovo tutte le equazioni, e risolverle, ma finora, se possibile, non aver parlato della Q.d.L. Finalmente dire che il risultato è identico ad avere usato la Q.d.L.; tracciare la genealogia di ogni merce (rispondendo alla domanda: perché L? perché non cavalli o carbone? risposta formale, unica quantità costante) e poi mostrare che il più semplice metodo è di sostituire, nelle equazioni, *r* con *S*. A questo punto soltanto dire che è Old Moor.[1]

'Old Moor' is Marx.

I. FIRST EQUATIONS

Let us begin with the well-known scheme of simple reproduction, as presented in *Capital,* Book II:[2]

Total du produit annuel:

I. $4000_c + 1000_v + 1000_{pl} = 6000$ moyens de production.

II. $2000_c + 500_v + 500_{pl} = 3000$ articles de consommation.

Valeur totale = 9000, excepté la partie du capital fixe qui continue à fonctionner sous son ancienne forme d'usage.[3]

[1] 'This manoeuvre is pivotal for the whole operation and everything depends on its success. We should proceed as follows. First, by developing the 1st equations, then the seconds (with *r*), then by introducing *w* as variable. This is the sensitive point. We can tell everything without divulging one secret: that the ratio between C and V+S is constant: and we can possibly say that the organic composition (expressed in vulgar terms) of the two groups is identical: and perhaps examine in detail the effects of a change in *r* or *w* on the prices of particular commodities. In any case, it is better to leave the clou for a later time.[...] Here the Crosscap comes in. By examining the Toy III, we notice a fixed ratio (between capital or part of it) and product, a ratio which is independent of *r*. Being careful to avoid highlighting other consequences, I notice that this gives a method (trick) to solve all (? at least I–III) the preceding equations. Focussing the attention on this method of solution: we can give to *r* an arbitrary value (i.e. 10% or 5% or 0%) thus linearizing the equations, solve them, get the "fixed ratio" from which we can derive the real *r,* and eventually solve the real equation. By this method, we can deal once more with all the equations, and solve them, but, up to now, without mentioning, if possible, the Q[uantities] o[f] L[abour]. Finally, we declare that this result is identical to that obtainable by using the Q.o.L.; trace the genealogy of each commodity (by answering the question: why L[abour]? why not horses or coal? The formal answer: it is the only constant quantity) and then show that the simplest method consists in substituting *S* for *r* in the equations. Now, and only now, we say that this is Old Moor'.

[2] Marx (1900), p. 444. For the reasons given above, n. 2, p. 28, all the quotations from *Capital,* Book II, refer to this French translation. Needless to say, in his library Sraffa also had the original German edition of 1885.

[3] 'Total annual product in commodities:

$4000_c + 1000_v + 1000_s = 6000$ means of production.

$2000_c + 500_v + 500_s = 3000$ articles of consumption.

Total value 9000, exclusive of the fixed capital persisting in its natural form, according to our assumption'.

In the scheme we find two industries, I and II, respectively producing means of production and consumption goods. We shall use, for short, the general term 'iron' to designate the first industry, and the general term 'wheat' to designate the second one.

The value of each product is divided by Marx into three components: cost of the means used up in production (in his original terminology: constant capital, identified by 'c'), labour cost (variable capital, identified by 'v') and profit (surplus value, identified by 'pl'). Wages, being supposed at the subsistence level, are entirely spent in buying wheat. The simple reproduction hypothesis implies that the profits realised by the two industries are entirely consumed and therefore also spent on wheat. The scheme is expressed in money terms: 'Les chiffres peuvent représenter des millions de marks, de francs ou de livres sterling'.[1]

As noted by Marx, the scheme implies an exchange between the two industries: 'Echange de I(v+pl) contre IIc'.[2] Industry I, in other words, has to buy wheat for 2000 million, giving iron in exchange for the same amount.

Sraffa's reference to this scheme as '1st equations' (the same expression, be it remembered, used to designate his own price equations without surplus) can be, at first sight, rather disconcerting. Leaving aside the existence of a surplus in Marx's scheme, the so-called 'equations' look like mere identities and, what is worse, identities expressed in money terms. With the apparently obvious consequence that they cannot tell us anything about the process of price determination.[3] But a simple reconsideration proves that this conclusion is quite misleading. The exchange between the two industries implies that 2000/6000 = 1/3 of the gross production of iron has to be exchanged at par with 2000/3000 = 2/3 of the gross production of wheat. These ratios—1/3 and 2/3—are physical ratios: between quantities, respectively, of iron and of wheat. If we assume gross production as the physical unit of measurement for products (as it is perfectly legitimate to do), these ratios become physical coefficients. And, on the basis of these physical coefficients, we can conclude that Marx's scheme implies a unique relative price of iron in terms of wheat: equal to 2. This is a very simple, but definitely not trivial, theory of prices.

However, this particular example has two weaknesses. First of all, it is, so to speak, too simple. Indeed, it so happens that the exchange ratio can be determined by the direct comparison between the excess productions of the two industries (1/3 versus 2/3). This could be interpreted, wrongly, as a particularly crude version of the usual supply and demand model. In fact, Sir Roy Harrod did not avoid this misunderstanding, in

[1] 'The figures may indicate millions of marks, francs or pounds sterling' (*ibid.*).

[2] 'I (v+s) *versus* IIc' (*ibid.*, p. 445).

[3] And indeed, when writing down these schemes, Marx was not concerned with the problem of relative values. He was primarily interested in investigating the physical proportions among industries, which were necessary for an orderly development of capitalist production. Proportions which could hardly be assured by the anarchy of the market. The very choice of the numbers used in his simple reproduction example, with the improbable assumption of a uniform organic composition of capital, is clearly due to his wish to set aside any value problem. On the contrary, Sraffa saw in the structure of the reproduction schemes, independent of the particular numbers chosen by Marx, important consequences for any theory of value. The Marxian tool was therefore used by Sraffa for a different purpose from that for which it was originally conceived, a not infrequent case in the history of economic analysis.

his review of *Production of Commodities* in the *Economic Journal*—a review which prompted the only public explanation ever written by Sraffa about his own theory.[1]

The adoption of a three-industry example, with the necessity of a triangular trade, is enough to avoid this misunderstanding. Marx himself offered such an example, with the division of the consumption goods into two sub-categories—workers' necessaries, and luxury goods for capitalists—together with the corresponding division of industry II into the two sub-industries, IIa and IIb. Unfortunately, his example cannot be used because of the second weakness, which we now address.

The second weakness is theoretical in nature: the coefficients are affected by very different causes. The requirements of iron, as well as wheat for workers (insofar as wages are kept to their minimum subsistence level) can be regarded as determined by purely technical factors. On the contrary, wheat consumption out of profits depends entirely on the arbitrary and provisional hypothesis of simple reproduction. This difficulty can be bypassed considering an economy without surplus (where wages are necessarily at their subsistence level, and consumption out of profits is obviously nil).

It must be concluded that the way out of both difficulties consists in using an example of a three-industry economy without surplus.

II. 'RIPRODUZ. SEMPLICE SENZA PLUSVALORE'[2]

Let us consider the very first equations written by Sraffa in winter 1927 (D3/12/5):
No surplus

$$A = a_1 + b_1 + c_1$$

$$B = a_2 + b_2 + c_2$$

$$C = a_3 + b_3 + c_3$$

where $A = \Sigma a$, $B = \Sigma b$, and $C = \Sigma c$.

Once again, this is a rather disconcerting way of writing. Equations without unknowns? Equations where heterogeneous quantities are summed up? Sraffa's formal education (classic high school and faculty of jurisprudence) had indeed a strong humanist bias, but it is also true that his papers attest, from the 1920s on, a good acquaintance with algebra. And we cannot reasonably suspect him of having committed such trivial errors. In fact, these 'equations' make sense only if we interpret them as a simple algebraic transcription (with letters substituted for numbers) of a Marxian scheme with three industries (simple reproduction is attested by the Σ conditions), and without surplus.

The quantities summed up are homogeneous, all being measured in money terms. The unknowns are not explicitly stated, but the scheme implies—as we have seen in the original Marxian scheme—a unique set of relative prices, which is determined by the necessary exchanges: $(A - a_1)$ versus $(b_1 + c_1)$ etc.

[1] Harrod (1961) and Sraffa (1962).
[2] 'Simple reproduction without surplus value'.

Now Sraffa's comments become perfectly comprehensible: 'These are homogeneous linear equations. They have infinite sets of solutions, but the solutions of each set are proportional. These proportions are univoche. These proportions we call ratios of Absolute Values. They are purely numerical relations between the things *A, B* . . .'. This last remark should not be overlooked: 'It was only Petty and the Physiocrats who had the right notion of cost as *the loaf of bread.* Then somebody started measuring it in labour, as every day's labour requires the same amount of food… A. Smith & Ricardo, & Marx indeed began to corrupt the old idea of cost,—from food to labour', Sraffa writes in November 1927 (D3/12/4).

Sraffa's peculiar way of writing could not but make professional mathematicians nervous. In a paper dated June 1928 (D3/12/2) Sraffa records observations made by Frank Ramsey, presumably about the above set of equations.

Equations without surplus: each quantity must be expressed by *two* letters, one being the number of units, the other the unit of the commodity. Otherwise, if I use only one letter, this would stand for heterogeneous things and the sum would be meaningless.

Thus, in order to write an ordered set of equations, Sraffa needed a new lettering system. Capital letters stand now for unit prices, and small letters for the number of units. However, the new equations are clearly nothing but the original ones rewritten following Ramsey's suggestion.

$$aA = a_1A + b_1B + c_1C$$
$$bB = a_2A + b_2B + c_2C$$
$$cC = a_3A + b_3B + c_3C$$

'Which is the unit?—asks Sraffa—*A*, *B*…? *a*? a_1?' (D3/12/5). We can choose both the physical unit for each commodity produced, and the price unit. It would be perfectly possible to assume, as in *Production of Commodities,* from §3 to 11, an arbitrary price (*A*, *B* or *C*) as unit. It would be equally possible to assume gross production as the physical unit for the three commodities, as we have done when dealing with the Marxian scheme, and therefore to set $a = b = c = 1$. But Sraffa, probably to avoid misunderstanding on the sensitive issue of returns, preferred not to do so.

Our interpretation is confirmed beyond reasonable doubt by the numerical examples devised by Sraffa to solve his equations in practice. See for instance the following document (D3/12/2):

									Values		
17A	=	2A	+	15B	+	20C			A	=	3B
28B	=	5A	+	7B	+	4C			B	=	1/3C
35C	=	10A	+	6B	+	11C			C	=	1/2A

On this example (which was revisited by Sraffa on the 20 February 1955) Sraffa makes a remarkable comment:

It is clear at once that these technical relations of production leave no room to play with: the values are rigidly fixed, & neither preferences nor . . . can have any influence unless they change these relations.

It may be noted that they do not represent only the cost of production: they equally show the use, or disposal, of each product.

The meaning and the relevance of the final sentence will possibly be clearer if we look back at the physical coefficients as they appeared in our treatment of Marx's simple reproduction scheme: quotas of a particular gross product allotted to the various industries.

III. 'RIPRODUZ. SEMPLICE CON PLUSVALORE TUTTO CONSUMATO'[1]

The insertion of a positive surplus into the price equations proved not particularly easy. There is a document (D3/12/2) with a two-product example (with surplus, triangular trade is no longer needed) which is particularly clear and meaningful.

Surplus only in A

$$10A + 4A = 3A + 9B$$

$$12B = 7A + 3B$$

There are two solutions. Why?

'Surplus only in A' means that surplus is entirely consumed and that the first industry obviously produces a consumption good (wheat). Note that this example is structurally identical to that of *Production of Commodities,* §5.

By solving the two equations separately, Sraffa finds two solutions (A/B = 9/11 and A/B = 9/7) and concludes: 'Le equazioni sono contraddittorie quindi non esiste alcuna soluzione'[2] (compare with *Production of Commodities,* §4: 'If the economy produces more than the minimum necessary for replacement and there is a surplus to be distributed, the system becomes self-contradictory'). In fact, he was discovering that, with a surplus, prices are influenced by its distribution. The two solutions found are the extreme solutions that limit the range of all economically viable prices: A/B equal to 9/11 attributes a zero profit to the first industry and a profit of 36/11 to the second industry, enabling its capitalist to buy the entire surplus, made up of four units of wheat. The reverse is true with A/B = 9/7. Every solution within these two extremes corresponds to a particular distribution of the surplus between the two industries.

IV. 'UGUALE A 2 IN FORMA DI 1'[3]

In order to have two independent and not contradictory equations, we need additional information about the distribution of the surplus between the industries. If, for example, one and three are the units of wheat-surplus respectively allotted to the two industries, we get the following system:

[1] 'Simple reproduction with surplus value entirely consumed'.
[2] 'The equations are contradictory and therefore there is no solution'.
[3] 'Equal to 2 in form of 1'.

$$14A = 3A + 9B + 1A$$
$$12B = 7A + 3B + 3A$$

which has a unique solution: A/B = 9/10 (intermediate between 9/11 and 9/7). In this way, we come full circle: indeed, these equations are formally equivalent to those corresponding to the original Marxian scheme.

V. A DETOUR: 'SURPLUS A SEPARATE INDUSTRY'

We are now able to follow another Marxian suggestion: the division of industry II (consumption goods) into two sub-industries, respectively producing workers' necessaries and luxury goods for capitalists. The idea is now that the excess production of iron and wheat are entirely used in a third industry to produce a luxury good ('silk', which constitutes the social surplus, devoted to capitalist consumption). Consider the following example, devised by Sraffa (D3/12/2):

$$11A = 3A + 9B$$
$$13B = 7A + 3B$$
$$S = 1A + 1B$$

these are contradictory, whether S is equal or not to zero.

Once again, the equations do not work because we lack some fundamental information, i.e., how the surplus (silk) is divided among industries. If, for example, we suppose that silk is equally distributed among the three capitalists, the system becomes perfectly determined:

$$11A = 3A + 9B + 1/3S$$
$$13B = 7A + 3B + 1/3S$$
$$1S = 1A + 1B + 1/3S$$

VI. 'RIPRODUZ. CON ACCUMULAZ. TOTALE… PROPORZIONALE IN TUTTE LE INDUSTRIE'[1]

In the same way as the 'first equations' were written starting from the Marxian simple reproduction scheme without surplus, the 'second equations' were written starting from the extended reproduction scheme with surplus totally accumulated proportionally in each industry (in Marx's 'first example'[2] of extended reproduction surplus is not totally accumulated, but every industry grows, after a while, at a common rate of 10%).

If the surplus is entirely reinvested, and all the industries grow proportionally, reproduction requires again a 'necessary' exchange: the gross production of one indus-

[1] 'Reproduction with total accumulation… proportional in every industry'.
[2] Marx (1900), pp. 568–573.

try must be exchanged at par with its means of production augmented by a common factor.

The first versions of the 'second equations' were written at the end of 1927. The following system (D3/12/7) was actually drawn in November 1931:

$$vA = (va_1 + b_1)r$$
$$B = (va_2 + b_2)r$$

where the variables are 'the value v of a in terms of b, and the rate of surplus r'.

In these equations, which are the mirror image of those corresponding to simple reproduction, prices are, once more, 'rigidly fixed' and the technical relations leave 'no room to play with'.

VII. ACCUMULATION LEAVES THE SCENE

However, once written, these equations are capable of an obvious double interpretation. The factor r can be considered as a common surplus, or growth, factor; but it can also be interpreted as a common profit factor. Needless to say, the two interpretations have very different economic implications. Let us only note that, while a common growth rate, in our decentralized economy, requires a common profit rate ('chi presterebbe a un'industria che non vende abbastanza per riprodursi?')[1] the reverse is not true. Moreover, the determination of a constant growth rate requires the assumption of constant returns, while a common profit rate does not.

In conclusion, the profit interpretation is equally powerful and, at first glance, much more general than its alternative growth interpretation. And this is presumably the reason why accumulation (whether total, proportional or accelerated) disappears in the 1930s from Sraffa's price equations

VIII. 'MY HYPOTHESIS'

In the early 1940s, Sraffa wrote down the 3rd equations and devised his fundamental standard commodity: i.e., the essential instruments for his 1960 book. What may be surprising is that he initially obtained these results thanks to a most un-Sraffian hypothesis, namely that product and means of production are physically homogeneous. In fact, when Sraffa adopted this hypothesis, he thought it was much more general and, when he realized that it implied the physical homogeneity between inputs and output, he abandoned it and spoke of a 'disastro del modello'.[2] The reconstruction offered here is largely speculative, but it has at least the advantage of offering a plausible and consistent interpretation of the papers concerned.

To begin at the beginning. The fundamental characteristic of an economic system is its physical productivity, i.e. its capacity to produce a surplus. This characteristic is

[1] 'who would lend capital to an industry which does not yield enough for its own reproduction?'.
[2] 'disaster of the model'.

summarised by a strategic index: the surplus rate (or factor). This index must depend only on technical factors and cannot be affected by merely economic or social factors, such as relative prices or income distribution. From this point of view, the transition from the uniform accumulation to the uniform profit interpretation for the 2nd equations allows a greater generality, but is not without consequences. The surplus rate is clearly incorporated in the original versions of the first two sets of equations.

In the 1st equations the surplus rate is zero by construction, as is shown by the fact that the social product is strictly identical to the set of the means of production advanced. In the 2nd equations in the total accumulation version, the surplus factor can be easily observed: (i) as the physical ratio of the product to the means of production advanced, and/or (ii) as the simple ratio of every commodity produced to its quantity used up in production.

What happens in the 2nd equations in the uniform profit version? The second way of computing the surplus rate becomes clearly impracticable. Indeed, the example with a surplus offered in *Production of Commodities,* §5, shows that a zero excess production of one commodity (iron) is perfectly compatible with a positive social surplus.

However, there is a remarkable feature. The same factor *r* represents, in the same set of 2nd equations, both the surplus factor—the physical ratio between social product and means of production used—and the (maximum) profit factor, i.e. the value ratio between gross income and non-wage capital advanced. An important conclusion, which Sraffa reasserted with emphasis in *Production of Commodities,* §22 (where the surplus rate is renamed balancing or Standard ratio) at the end of a particularly laborious chapter, where no mention is made of accumulation.

From this undisputable fact, i.e. the equality between the physical surplus rate and the (maximum) profit rate, Sraffa draws his conclusion or, better, 'hypothesis'. Gross income and capital are collections of commodities weighted or averaged by means of prices; prices are influenced by income distribution; however, on average, the ratio between gross income and non-wage capital remains constant and equal to the surplus rate.

After *Production of Commodities,* we are now able to recognise the fallacy implied in this conclusion, but this was not a trivial error 60 years ago (as is shown by the fact that it was repeated by Samuelson 20 years later).[1] Moreover, it proved to be a fruitful error.

IX. THIRD EQUATIONS

In 1942 Sraffa was eventually able to write down his 3rd equations in a quasi final version, very similar to that familiarized by *Production of Commodities* (apart from wage, which is still considered as part of the capital and therefore paid *ante factum)*. Reproduced here are the equations in a simplified form, with only two industries. The symbols are those later used in the 1960 book.

[1] Samuelson (1962).

$$(A_a p_a + B_a p_b + L_a w)(1 + r) = A p_a$$
$$(A_b p_a + B_b p_b + L_b w)(1 + r) = B p_b.$$

Sraffa is particularly interested in the (theoretical, much more than algebraic) procedure for solving this set of equations, as well as their more straightforward antecedents. And we can now understand the procedure he envisaged in his tactical-expositional programme written on 21 August 1942.

The idea is the following. If 'gross product' and 'non-wage capital' are composite commodities whose ratio is unaffected by prices and therefore by r (by 'Hypothesis') we can proceed as follows. We can give to r an arbitrary value: in particular, we can set $r = 0$. This makes the equations linear (they become structurally identical to the 1st equations) and we can easily compute relative prices, which happen to be proportional to the quantities of labour embodied in the various commodities ('Q.o.L.'). We are now able to compute the maximum rate of profit R (equal to the surplus rate) as the ratio of the value of net production to the value of non-wage capital. At this point, everything becomes clear and transparent. Indeed, we can write

$$(C + Lw)(1 + r) = C(1 + R)$$

where C stands for non-wage capital (constant capital). Assuming the value of the net product and the total quantity of labour employed as unit measure for prices and labour respectively, we get

$$r/R + wr + w = 1$$
$$r = R(1 - w(1 + r)).$$

On the same day, 5 May 1943, when Sraffa obtains this result, he notices that, with 'wages paid from product' (*ex post*), this 'fundamental equation' becomes even simpler. In this case we have:

$$C(1 + r) + Lw = C(1 + R).$$

After the suitable normalizations, the two distributive variables are shown to be linked by a very simple linear inverse relation:

$$r = R(1 - r).$$

XI. 'DISASTRO DEL MODELLO'[1]

But, as Sraffa rapidly realized, the Hypothesis was so restrictive as to be practically useless. In fact, it amounted to assuming an economic system in which only one commodity is produced: precisely that composite commodity which is produced in the total accumulation equation.

Happily enough, the 'disaster' became evident when Sraffa was working with the 3rd equations, a circumstance which allowed a radical shift in perspective, diverting

[1] 'Disaster of the model'.

attention from what was actually produced in the system to the choice of the unit used by the analyst to measure wages and prices.

XII. 'HYPOTHESIS' AND 'TRICK': AN ALGEBRAIC SUMMING-UP

The whole argument can be summarised by using the conventional notation of linear models. The original examples inspired by Marx's schemes (simple reproduction without surplus; extended reproduction with total and proportional accumulation) share the restrictive assumption that the vector $\boldsymbol{x}$ of the quantities produced be proportional to the vector $\boldsymbol{xA}$ of the means of production used. In other terms, the two vectors represent different quantities of the same composite commodity.

$$(1 + S)\boldsymbol{xA} = \boldsymbol{x}$$

where $(1 + S)$ is the proportionality factor between $\boldsymbol{xA}$ and $\boldsymbol{x}$. S is therefore the surplus rate, equal to zero in the 1st equations and positive and uniform in the 2nd equations.

The price equations are:

$$(1 + R)\boldsymbol{Ap} = \boldsymbol{p}$$

where $(1 + R)$ is a uniform profit factor. The two equations imply that $S = R$.

We can write:

$$(1 + R)\boldsymbol{xAp} = \boldsymbol{xp}$$

where $(1+ R)$ is the value ratio between the product and the means of production. R is the maximum rate of profit, corresponding to the minimum (subsistence) wage. R, which is a ratio between values (net income and capital) is equal to S, which is a physical ratio between quantities. From this equality follows the 'hypothesis' that the value ratio between product and capital be constant and equal to the surplus rate, given by technology. Since prices are influenced by the profit rate, which can be less than R, we are forced to conclude that the value ratio of social product to capital is not affected by price variations.

Now we come to the 3rd equations, where the actual production levels are set equal to 1, i.e., $\boldsymbol{x} = \mathbf{1}$:

$$(1 + r)\boldsymbol{Ap} + \boldsymbol{l}w = \boldsymbol{p}$$

The 'trick' used to solve these equations, according to the 'hypothesis', is the following:

(i) solve the (linear) equations for relative prices $\boldsymbol{p}^*$ corresponding to $r = 0$;
(ii) compute the maximum profit factor (equal to the surplus factor) using these prices: $(1 + R) = \mathbf{1}\boldsymbol{p}^*/\mathbf{1}\boldsymbol{Ap}^*$;
(iii) by setting $\mathbf{1}\boldsymbol{l}$ and $\mathbf{1}\boldsymbol{ApR}$ equal to 1, derive the fundamental relation $r = R(1 - w)$;
(iv) compute actual prices corresponding to a certain level of w (or r);

(v) note that prices $\boldsymbol{p}^*$ are proportional to labour contents and that therefore the rate of profit is equal to the ratio of Marxian surplus value to constant capital.

Afterwards, Sraffa realized that his hypothesis amounted to the very restrictive assumption of a rigid proportionality among the components of $\mathbf{1}$ and $\mathbf{1}\boldsymbol{A}$, a true disaster for the model.

XIII. 'TOY III'

Nonetheless, let us imagine the 3rd equations corresponding to the vector $\boldsymbol{x}$ of quantities produced; where $\boldsymbol{x}$, be it remembered, is equal to $\boldsymbol{xA}(1 + S)$. In this case, the equation would become:

$$(1 + r)\boldsymbol{xAp} + w\boldsymbol{xl} = \boldsymbol{xp}$$

If we set $S\boldsymbol{xAp} = 1$ and $\boldsymbol{xl} = 1$, we get:

$$r = S(1 - w)$$

This peculiar result is not due to the hypothetical $\boldsymbol{x}$ activity levels adopted, but only to our special normalizations. The vector $\boldsymbol{x}$ is fundamental, but only in setting the unit used to measure wages and prices. And $\boldsymbol{x}$ is easily obtainable, once the value of S is known, which is equal to the value of R: the profit rate corresponding to $w = 0$.

We can therefore recycle the original trick in a new version:

(i) compute R in the 3rd equations as the value of r corresponding to $w = 0$;
(ii) compute the corresponding vector $\boldsymbol{x} = \boldsymbol{xA}(1 + R)$;
(iii) set the suitable normalizations in terms of $\boldsymbol{x}$;
(iv) compute r, given w (or vice versa);
(v) compute relative prices $\boldsymbol{p}$.

Old Moor remains behind the scenes.

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